

**Class Notes for BCH 6746 (Spring, 2000)**  
Structural Biology: Theory and Applications of NMR Spectroscopy

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## **Week 1: Introduction to the basics: Bloch equations**

**References:** Most NMR books. These notes were constructed from combinations of:  
“Protein NMR spectroscopy: Principles and Practice” Cavanagh, Fairbrother, Palmer, and Skelton, Academic Press, (1996).  
“Biomolecular NMR Spectroscopy” Evans, Oxford University Press, (1995).  
“Principles of Nuclear Magnetic Resonance in One and Two Dimensions” Ernst, Bodenhausen, and Wokaun, Oxford (1987).

### **Nuclear spin and angular momentum**

Some atomic nuclei have an intrinsic property called “spin”. This was first demonstrated in 1922 by the Stern-Gerlach experiment, in which a beam of silver atoms were passed through a magnet field and split into two beams. These two beams represent the two states,  $\alpha$  and  $\beta$ , of the silver (spin 1/2) nuclei.

The nuclear spin has an intrinsic angular momentum, a vector that is represented by the symbol  $\mathbf{I}$  (vectors will be in bold). Vectors have 3 orientations (x, y, and z) and a length. However, the Heisenberg Uncertainty Principle tells that we can only know one orientation and the length simultaneously. By convention in NMR, we say that we know the z-orientation of the angular momentum of a nucleus in a magnetic field. The square of the magnitude of  $\mathbf{I}$  is given by

$$|\mathbf{I}^2| = \hbar^2 [I(I+1)] \quad (1-1)$$

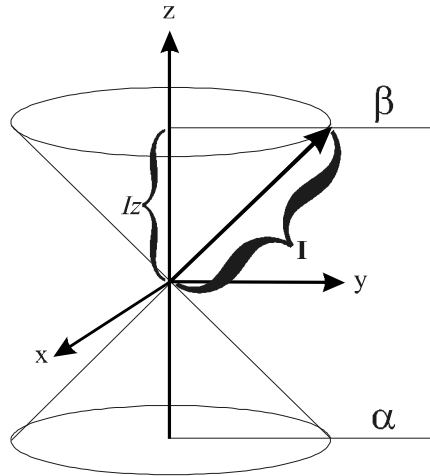
where  $I$  is the spin quantum number (e.g. 1/2) and  $\hbar$  is Planck’s constant divided by  $2\pi$ . The z-component of  $\mathbf{I}$  is given by

$$I_z = \hbar m \quad (1-2)$$

where  $m$  is the quantum number with values  $m = (-I, -I+1, \dots, I-1, I)$ . In the case of spin 1/2 nuclei,  $I = 1/2$  and  $m$  can be -1/2 and 1/2.

A diagram illustrating these principles is shown below. In the diagram, the coordinate system has been placed in the center of two cones. The angular momentum vector  $\mathbf{I}$  is shown on the edge of the

upper cone. The projection of  $\mathbf{I}$  onto the z-axis is  $I_z$ , but  $\mathbf{I}$  is completely undetermined in the x,y plane.



Nuclei that have spin angular momentum also have a magnetic moment given by

$$\mathbf{m} = \gamma \mathbf{I} \quad (1-3)$$

where  $\gamma$  is the gyromagnetic ratio (sometimes called the magnetogyric ratio).  $\gamma$  is a physical property of each nucleus. For a given abundance, nuclei with higher values of  $\gamma$  produce higher sensitivity NMR spectra. The z-component of the magnetic moment is thus given by

$$\mu_z = \gamma I_z = \gamma \hbar m \quad (1-4)$$

## Nuclear energies in a magnetic field

The energy of the magnetic moment of a nuclear spin in a magnetic field is given by

$$E = - \mathbf{m} \cdot \mathbf{B} \quad (1-5)$$

where  $\mathbf{B}$  is the magnetic field (in general  $\mathbf{B}$  is a vector quantity). To conduct an NMR experiment, a sample is first placed into a static magnetic field. By convention, the direction of the static magnetic field is along the z-axis, and the magnitude of the magnetic field is given by  $B_0$  (no longer a vector since it points only along the z-axis). In this case, the energy of a nuclear spin in an NMR magnet becomes

$$E = - \mu_z B_0 = -\gamma I_z B_0 \quad (1-6)$$

and the energy associated with a particular quantum number  $m$  is

$$E_m = -m\hbar\gamma B_0 \quad (1-7)$$

NMR transition energies are very small. These small energies translate into low sensitivity. In the absence of a magnetic field, the  $\alpha$  and  $\beta$  states are equally populated, leading to no net magnetization (that is why we need large static magnets to do NMR). When samples are placed into a magnetic field, a *small excess* of nuclei fall into the  $\alpha$  state. This excess of spins in the  $\alpha$  over the  $\beta$  states accounts for the entire net magnetization which is used in the NMR experiment. The ratio of the number of spins in the  $\alpha$  state to those in the  $\beta$  state is given by a Boltzmann distribution

$$\frac{N_{\alpha}}{N_{\beta}} = e^{\frac{\Delta E}{k_B T}} \quad (1-8)$$

where  $\Delta E$  is the difference of energies of the  $\alpha$  and  $\beta$  states,  $k_B$  is Boltzmann's constant, and  $T$  is the absolute temperature. Typical values of magnetic field strengths produce as little as 1 in 10,000 excess  $\alpha$  spins over  $\beta$ . Higher magnetic fields produce correspondingly larger differences in spin states, leading to greater sensitivity.

Quantum mechanical selection rules dictate that NMR signals can only arise when  $m$  changes  $\pm 1$ . From Planck's law ( $E = h \nu$ ) the frequency ( $\nu_0$ ) of an NMR transition in a magnet field  $B_0$  is

$$\nu_0 = \frac{E}{h} = \frac{\gamma B_0}{2\pi} \quad (1-9)$$

The units of  $\nu_0$  are radians/second. More commonly, NMR frequencies are expressed in Hertz (Hz) from  $\nu_0 = \omega_0/2\pi$ , yielding the well-known expression

$$\omega_0 = \gamma B_0 \quad (1-10)$$

## Behavior of bulk magnetization in a magnetic field

We will now consider "bulk" magnetization  $\mathbf{M}$  that arises from all of the magnetic moments in a sample.  $\mathbf{M}$  experiences a torque when placed in a magnetic field according to

$$\frac{d\mathbf{J}(t)}{dt} = \mathbf{M}(t) \times \mathbf{B}(t) \quad (1-11)$$

where  $\mathbf{J}(t)$  is the bulk spin angular momentum. Note that all three vector quantities in Equation 11 are time dependent. The time-dependence of the magnetic field comes about when we apply radio frequency (*rf*) pulses along the x- or y-axis (described below). Equation 11 is essentially identical to an equation that describes the motion of a gyroscope (see, for example, Goldstein "Classical Mechanics" Addison-Wesley, 1980):

$$\frac{d\mathbf{L}(t)}{dt} = \mathbf{r} \times m\mathbf{g} \quad (1-12)$$

where  $\mathbf{L}(t)$  is the gyroscope's angular momentum,  $\mathbf{r}$  the radius from the fixed point of rotation,  $m$  the mass, and  $\mathbf{g}$  gravity. Thus, a nuclear spin in a magnetic field will behave much like a gyroscope in a gravitational field. To make Equation 11 more useful, we use the relationship given in Equation 4 and multiply each side by  $\gamma$  to yield

$$\frac{d\mathbf{M}(t)}{dt} = \mathbf{M}(t) \times \gamma \mathbf{B}(t) \quad (1-13)$$

Equation 13 is the basis of the Bloch equations.

## Bloch equations

In 1946 Felix Bloch formulated a set of equations that describe the behavior of a nuclear spin in a magnetic field under the influence of *rf* pulses. He modified Equation 13 to account for the observation that nuclear spins “relax” to equilibrium values following the application of *rf* pulses. Bloch assumed they relax along the z-axis and in the x-y plane at different rates but following first-order kinetics. These rates are designated  $1/T_1$  and  $1/T_2$  for the z-axis and x-y plane, respectively.  $T_1$  is called spin-lattice relaxation and  $T_2$  is called spin-spin relaxation. Both of these will be described in more detail later in the class. With the addition of relaxation, Equation 13 becomes

$$\frac{d\mathbf{M}(t)}{dt} = \mathbf{M}(t) \times \gamma \mathbf{B}(t) - \mathbf{R}(\mathbf{M}(t) - M_0) \quad (1-14)$$

where  $\mathbf{R}$  is the “relaxation matrix”. Equation 14 is best understood by considering each of its components:

$$\begin{aligned} \frac{dM_z(t)}{dt} &= \gamma[M_x(t)B_y(t) - M_y(t)B_x(t)] - \frac{M_z(t) - M_0}{T_1} \\ \frac{dM_x(t)}{dt} &= \gamma[M_y(t)B_z(t) - M_z(t)B_y(t)] - \frac{M(t)_x}{T_2} \\ \frac{dM_y(t)}{dt} &= \gamma[M_z(t)B_x(t) - M_x(t)B_z(t)] - \frac{M(t)_y}{T_2} \end{aligned} \quad (1-15)$$

The terms in Equation 15 that do not involve either  $T_1$  or  $T_2$  are the result of the cross product in Equation 14. Equation 15 describes the motion of magnetization in the “laboratory frame”, an ordinary coordinate system that is stationary. Mathematically (and conceptually) the laboratory frame is not the simplest coordinate system, because the magnetization is moving at a frequency  $\omega_0 = \gamma B_0$  in the x-y (transverse) plane. A simpler coordinate system is the “rotating frame”, in which the x-y plane rotates around the z-axis at a frequency  $\Omega = -\gamma B_0$ . In the rotating frame, magnetization “on resonance” does not precess in the transverse plane. The transformation of Equation 15 to the rotating frame is achieved by replacing each  $B_z$  (defined as  $B_0$ ) by  $\Omega/\gamma$ :

$$\begin{aligned}
\frac{dM_z(t)}{dt} &= \gamma[M_x(t)B_y^r(t) - M_y(t)B_x^r(t)] - \frac{M_z(t) - M_0}{T_1} \\
\frac{dM_x(t)}{dt} &= -\Omega M_y(t) - \gamma M_z(t)B_y^r(t) - \frac{M_x(t)}{T_2} \\
\frac{dM_y(t)}{dt} &= \gamma M_z(t)B_x^r(t) + \Omega M_x(t) - \frac{M_y(t)}{T_2}
\end{aligned} \tag{1-16}$$

In Equation 15, the components of  $\mathbf{B}$  have been written with r superscripts to denote that it is a rotating frame. From this point onward, the rotating frame will be assumed without the superscript.

### Physical interpretation of Bloch Equations: Single pulse experiment

We will now examine the behavior of Equation 16 under two different limiting conditions, the effect of a short *rf* pulse and free precession. The *rf* pulse will be assumed to be very short compared to either relaxation times  $T_1$  and  $T_2$  as well as the angular frequency  $\Omega$ . This assumption is valid for many typical pulsed NMR experiments in which the pulse lengths can be as short as 5  $\mu$ s. We will apply the *rf* pulse along the x-axis. These conditions allow us to neglect terms in Equation 16 that contain  $T_1$ ,  $T_2$ ,  $\Omega$ , and  $B_y$ .

$$\begin{aligned}
\frac{dM_z(t)}{dt} &= -M_y(t)\gamma B_x^r(t) \\
\frac{dM_x(t)}{dt} &= 0 \\
\frac{dM_y(t)}{dt} &= M_z(t)\gamma B_x^r(t)
\end{aligned} \tag{1-17}$$

Before solving Equation 17, it is worth a short digression to discuss the meaning of  $B_x(t)$  and  $B_y(t)$ . Recall that  $B_0$  is the static magnetic field strength oriented along the z-axis.  $B_x(t)$  and  $B_y(t)$  are magnetic fields oriented along the x- and y-axes that are generated by *rf* pulses. By analogy to  $\omega_0 = \gamma B_0$  defining the frequency of the NMR transitions in the static magnetic field, we can see that the terms  $\gamma B_x(t)$  and  $\gamma B_y(t)$  are frequencies of the magnetization rotating around the x- or y- axis. Thus, applying these frequencies for different periods of time will allow for different degrees of rotation around the x- or y-axis. If we introduce a frequency of rotation about the x-axis as  $\omega_x = \gamma B_x(t)$ , solutions to Equation 17 are

$$\begin{aligned}
M_z(t) &= M_0 \cos(\omega_x t) \\
M_x(t) &= 0 \\
M_y(t) &= M_0 \sin(\omega_x t)
\end{aligned} \tag{1-18}$$

Finally, if we let  $\alpha = \omega_x t$  be the pulse angle, Equation 18 shows that application of a magnetic field (*rf* pulse) along the x-axis causes the magnetization which was originally along the z-axis to rotate toward the y-axis by an angle  $\alpha$ . Note that when  $\alpha = 0$ ,  $M_z(t) = M_0$  and  $M_y(t) = 0$  (all the magnetization is still pointing along the z-axis). When  $\alpha = 90^\circ$ ,  $M_y(t) = M_0$  and  $M_z(t) = 0$  (all the

magnetization is still pointing along the y-axis). Thus we have described the effects of a simple *rf* pulse.

More general solutions to Equation 16 can be written as series of rotations of starting magnetization by various angles and along various axes. An example of this in matrix notation for the x-pulse that we just described is

$$\mathbf{M}(t) = \mathbf{R}_x(\mathbf{a})\mathbf{M}_0$$

or

$$\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & -\sin a \\ 0 & \sin a & \cos a \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_0 \end{bmatrix} \quad (1-19)$$

The careful student might notice that the sign of rotation between Equations 19 and 18 are different. This stems from the fact that use of  $\omega_0 = \gamma B_0$ . Note that in these notes, there is a change in sign from Equations 7 and 10. This is a convention that was adopted by NMR spectroscopists, but is the wrong sign for rotation (see the books by Ernst and coworkers and Evans in the reference list).

The second limiting condition for Equation 16 is free precession in the absence of any applied pulse. In that case,  $B_x$  and  $B_y$  are both equal to zero, and Equation 16 becomes

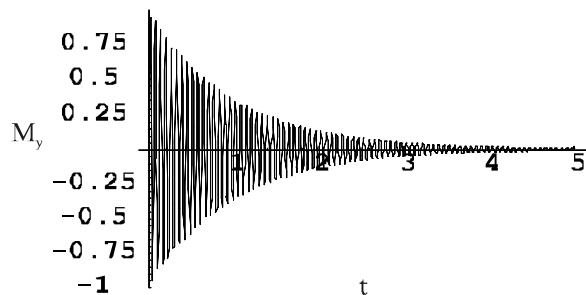
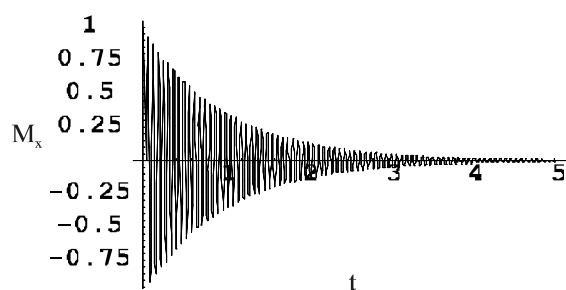
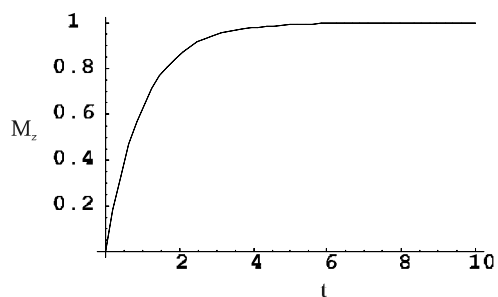
$$\begin{aligned} \frac{dM_z(t)}{dt} &= -\frac{M_z(t) - M_0}{T_1} \\ \frac{dM_x(t)}{dt} &= -\Omega M_y(t) - \frac{M_x(t)}{T_2} \\ \frac{dM_y(t)}{dt} &= \Omega M_x(t) - \frac{M_y(t)}{T_2} \end{aligned} \quad (1-20)$$

Students can verify that solutions to Equation 20 are

$$\begin{aligned} M_z(t) &= M_0(1 - e^{-\frac{t}{T_1}}) \\ M_x(t) &= M_0 \cos(\Omega t) e^{-\frac{t}{T_2}} \\ M_y(t) &= M_0 \sin(\Omega t) e^{-\frac{t}{T_2}} \end{aligned} \quad (1-21)$$

Equation 21 describes magnetization processing in the x-y plane at a frequency  $\Omega$  while it is relaxing along the z-axis at a rate  $1/T_1$  and relaxing in the x-y plane at a rate  $1/T_2$ .

Using values of  $M_0 = 1$ ,  $T_1=T_2=1$  s, and  $\Omega=100$  Hz, the following plots are obtained from Equation 21. The plots show the relaxation of magnetization back to the z-axis, and the familiar free induction decay (FID) in the x-y plane.



Expansions of  $M_x(t)$  and  $M_y(t)$  show that these are 90 degrees out of phase with one another, giving rise, after Fourier Transformation, to the real and imaginary parts of the NMR spectrum.

