## A few words about imaginary numbers and electronics

While most of us have seen imaginary numbers in high school algebra, the topic is ordinarily taught in abstraction without any kind of meaningful application. At least in my case, the algebra textbook was not very creative in convincing me that the there was anything of *real* value about an *imaginary* concept. The fact is, however, that algebra of complex numbers is pervasive in engineering of all sorts and very much so in signals and signal processing. Assuming that you may have chosen to ignore, or to forget, much of that material, here are a few high points in review.

Math is formal, and as long as we can make the rules self-consistent, there is no special requirement that we can use it to make direct measurement of the outside world. For example, many people take 2 or 3 kids to school. It is sensible to talk of taking 0 kids to school, but it really quite different than the choice of 2 or 3. Taking -2 kids to school makes sense only in a formal way. Yet it is both comfortable and practical to deal with negative numbers.

The idea for complex numbers starts with the observation that the square of an ordinary (real) number is always defined to be a positive number, because both a positive times a positive and a negative times a negative are *defined* to be positive. While the product of a negative and a positive number is negative, the two numbers making up that product must be different (because their signs are different) and therefore the negative number cannot be a perfect square. None of this stops us, however, from *defining* anew a number whose perfect square is negative. Therefore, mathematicians created a number, $i$, whose square is equal to -1:

$$i = \sqrt{-1}$$

The number, $i$, is called imaginary, but it is really no more or less imaginary than -2, or 0, as it is a formally defined concept, not a tangible number for counting for example. An important practical note, is that the variable name, $i$, was adopted by electrical engineers and physicists to refer to electrical current. This created an uncomfortable naming conflict. In most engineering disciplines and some physics contexts, the letter, $j$, is used instead:

$$j = \sqrt{-1}\ .$$

In the hope of minimizing long term confusion, the notes that follow use the latter convention.

We can use ordinary rules of algebra to extend this idea to the square root of other negative numbers, noting that the square of the product of two numbers, $(ab)^2 = a^2b^2$. Thus, the square of $aj$ is $a^2j^2$, but $j^2$ is -1, therefore $(ab)^2$ is equal to $-a^2$, and therefore $\sqrt{-a^2} = ja$. Equivalently, $\sqrt{-c} = j\sqrt{c}$. By definition, the product $ja$ is an imaginary number as well.

Complex numbers are defined as being the sum of a real number and an imaginary number, as in $a + jb$. We consider them as having a real and an imaginary part. The rules of arithmetic for complex numbers are nearly identical to those of the real numbers, except that we must keep track separately of the real and imaginary parts:

Let $F = a + jb$ and $G = c + jd$.

$$F + G = a + c + jb + jd = (a + c) + j(b + d).$$

$$F - G = a - c + jb - jd = (a - c) + j(b - d).$$

$$\begin{aligned} FG &= (a + jb)(c + jd) = ac + jbc + jad + (jb)(jd) \\ &= ac - bd + jbc + jad = ac - bd + j(bc + ad). \end{aligned}$$

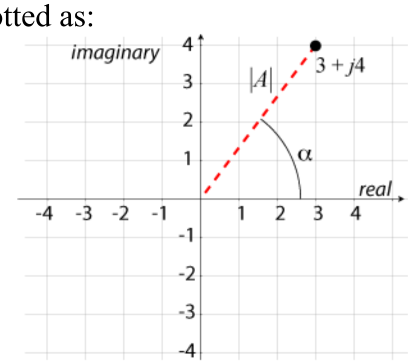
It follows, therefore, that:

$$
\begin{aligned}
F \div G &= \frac{(a+jb)}{(c+jd)} \\
&= \frac{(a+jb)}{(c+jd)}\frac{(c-jd)}{(c-jd)} \\
&= \frac{(a+jb)(c-jd)}{(c^2+d^2)} \\
&= \frac{ac+bd+jbc-jad}{(c^2+d^2)} \\
&= \frac{(ac+bd)}{(c^2+d^2)} + j\frac{(bc-ad)}{(c^2+d^2)}.
\end{aligned}
$$

You should be able to verify quickly that if $b$=$d$=0 then $F$ and $G$ are both real and that normal arithmetic rules apply: $F/G = a/c$. All real numbers are considered complex, with an imaginary part equal to 0$j$.

*How large are complex numbers?*

In grade school we learn to think about magnitude using the number line. One interpretation of this is that the numbers can be compared by their distance from 0; thus, -3 has the same magnitude as +3, leading to the concept of *absolute value*. We can think of complex numbers in an analogous manner. Since the real and imaginary parts of complex numbers are treated separately, we can consider them as two dimensional numbers. Geometrically, it is common to present complex numbers on a two-dimensional plane with real and imaginary axes. Thus, the number $A$=3+$j$4 constitutes an ordered pair, (3, 4), that can be plotted as:




(*figure 1*)

By analogy to the number line, we can consider the magnitude of that complex number as the distance from it to the origin (the length of the dashed line in the figure). Specifically, the magnitude of 3+$j$4, denoted with the absolute value symbol, is $|A| = |3 + j4| = \sqrt{3^2 + 4^2} = 5$. In general, the magnitude of a complex number is calculated by squaring the real and imaginary parts, adding these together and taking the square root – it is no more than the Pythagorean theorem. This implicitly suggests, by the way, that the magnitude of $|j| = 1$. Is this reasonable? Sure, because $j^2$=-1, whose absolute value is 1.

You might also want to verify for yourself that simple arithmetic operates just as it should. For example:

$$
\begin{aligned}
2A &= 2(3+j4) \\
&= 6+j8 \\
|2A| &= \sqrt{6^2+8^2} \\
&= 10.
\end{aligned}
$$

The magnitude is twice as large as before. You can do this for addition, subtraction, multiplication, etc…

Note also there is a natural translation here into polar coordinates. Instead of expressing a complex number as an ordered pair, we can, and often do, treat it as magnitude and phase, where the phase is the angle, α, it makes with the real (axis). The real part is therefore equal to $|A|\cos(\alpha)$ and the imaginary part is equal to $|A|\sin(\alpha)$.

We see elsewhere, that imaginary numbers have a deep role in laplace and Fourier transforms, that derives from the relation, attributed to Euler, that:

$$e^{jx} = \cos(x) + j\sin(x).$$

Deriving the Euler relation requires calculus, but I have restricted this note to algebra only. Notice that in the Euler relation, the real part of $e^{jx}$ is cos(x) and the imaginary part is $j$sin(x). The symmetry with the geometric representation above should be obvious.

*Physical Measurements of Complex Signals*

Although complex numbers contain imaginary parts, it is still possible to make measurements of complex quantities relatively easily by measure the magnitude. As an example, we can create a signal with the value $\mathrm{Re}(Ae^{j\omega t})$, which is by definition real (the notation, Re($x$) means the real part of $x$). As you can see from the Euler relation, this is equal to $A\cos(\omega t)$, a sinusoid.

Although this is not derived formally here, if we inject this signal into a linear device, such as an electronic amplifier, the output will be a scaled version, $B\cos(\omega t)$, where $B$ itself may be a complex number (even if $A$ was not). If we use an instrument such as a voltmeter or oscilloscope to measure the output of our device, the signal we receive will be a sinusoid of magnitude $|B|$. If $B$ is complex, the output waveform will have a phase shift and appear as $|B|\cos(\omega t + \varphi)$, where $\varphi$ denotes the phase. In general, then, a *linear* device driven by a sinusoid will create a scaled and phase-shifted output signal. Note that the scaling may depend on the frequency, ω. Ironically, although engineering math almost always represents angles in radian measure, the phase of a signal is usually measured in degrees. Go figure.