Bayesian Statistics in Neuroimaging

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Bayesian Inference

- Most statistical methods covered in introductory statistics courses are so called frequentist (or classical) methods.

- Bayesian inference is another approach that provides a somewhat different perspective.

- Bayesian methods have recently received a great deal of attention in fMRI research.

Classical vs Bayesian Approach

- The frequentist (or classical) point of view:
  - Probability refers to limiting relative frequencies.
  - Parameters are fixed unknown constants. Because they do not fluctuate, no useful probability statements can be made about them.
  - Statistical procedures should be designed to have well-defined long run frequency properties.

Classical vs Bayesian Approach

- The Bayesian point of view:
  - Probabilities describe a degree of belief.
  - Probability statements can be made about parameters, even though they are fixed constants.
  - Inferences are made about a parameter [illegible] by producing a probability distribution for it.

The Bayesian Method

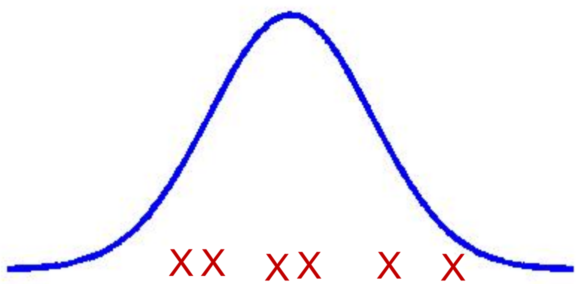
- Choose a probability density $p(\theta)$, the prior distribution, that expresses our beliefs about a parameter θ before we see any data.

- Choose a statistical model $p(y|\theta)$, the likelihood, that reflects our belief about y given θ.

- After observing y, update our beliefs and calculate the posterior distribution $p(\theta|y)$.

Example

- Suppose we observe a sequence of observations y_1, ….y_n from a $N(\mu,\sigma^2)$ distribution, where μ is unknown and σ^2 known.



Assume μ is the task-induced change in brain activity and y_i is the equivalent contrast for subject i.

Likelihood: $p(y_1,\ldots.y_n|\mu) = N(\mu,\sigma^2)$

- We are interested in estimating the parameter μ.

Prior Distribution

- In the Bayesian approach $\square$ can be described by a probability distribution. (Prior Distribution)

- The prior distribution is a subjective distribution, based on the experimenter's belief and is formulated prior to viewing the data.

- In our example assume $p(\square) = N(\square_0, \square_0^2)$

Posterior Distribution

- After a sample is taken from a population, the prior distribution can be updated using the information contained in the sample.

- The updated prior is called the posterior distribution. Updating is done using Bayes Rule:

$$p(\theta \mid y) = \frac{p(y \mid \theta)p(\theta)}{p(y)}$$

- Note that p(y) does not depend on θ.

- Hence, the posterior density is often written:

$$\underbrace{p(\theta \mid y)}_{\text{Posterior}} \propto \underbrace{p(y \mid \theta)}_{\text{Likelihood}}\, \underbrace{p(\theta)}_{\text{Prior}}$$

Posterior Inference

- The posterior distribution contains all current information about the parameter [illegible].

- Numerical summarizes (e.g., mean , median, mode) of the distribution are used to obtain point estimates of the parameter.

- We can also make probability statements about the parameter of interest and create posterior intervals.

Example

- Let $y_1,\ldots,y_n$ be observations from a $N(\theta,\sigma^2)$ distribution, with θ unknown and σ^2 known.

- Suppose we take the prior distribution of θ to be $N(\mu_0,\tau_0^2)$ for some choice of μ_0 and τ_0^2.

$$p(\theta \mid y_1,\ldots y_n) \propto p(y_1,\ldots y_n \mid \theta)p(\theta)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2\right\}\exp\left\{-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right\}$$

- It can be shown that

$$\theta \mid y_1, \ldots y_n \sim N(\mu_n, \tau_n^{\,2})$$

where

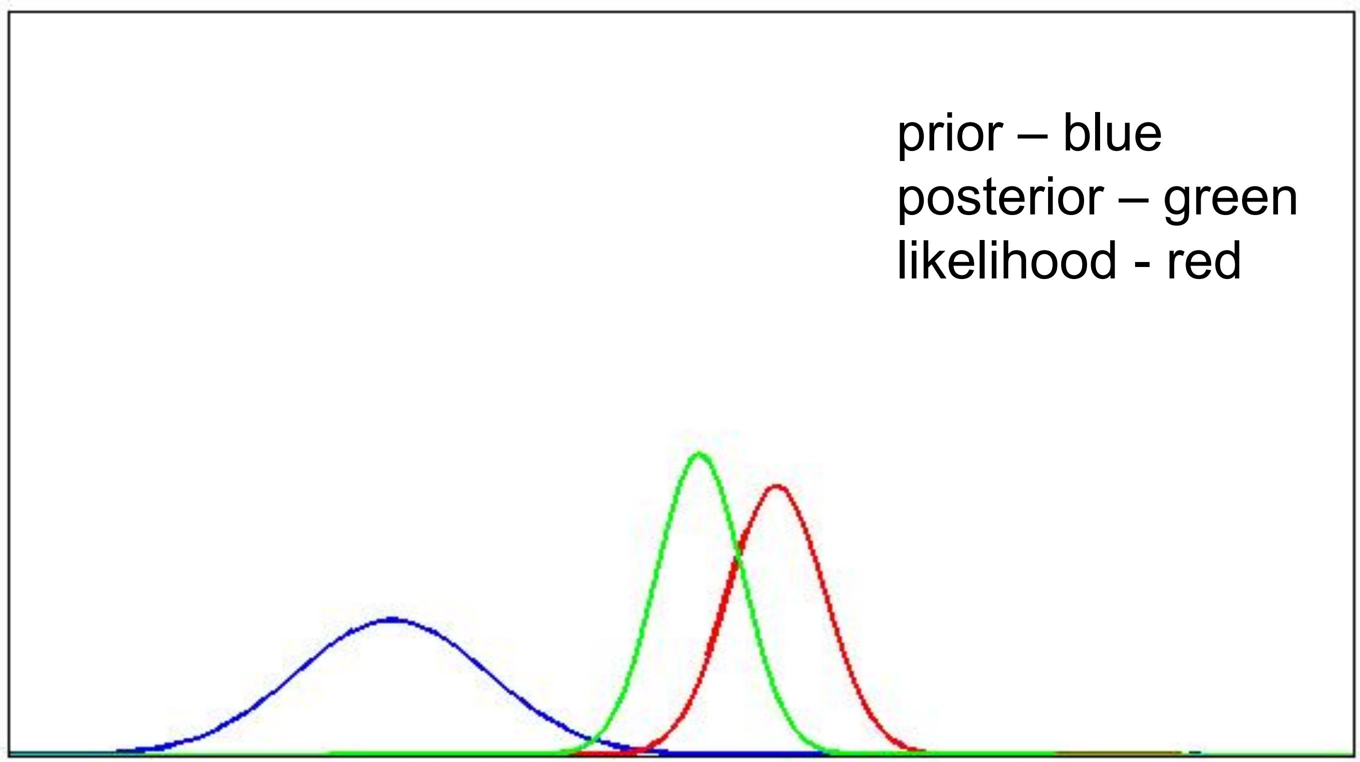
$$\mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

- Note: $\mu_n = w\mu_0 + (1-w)\bar{y}$

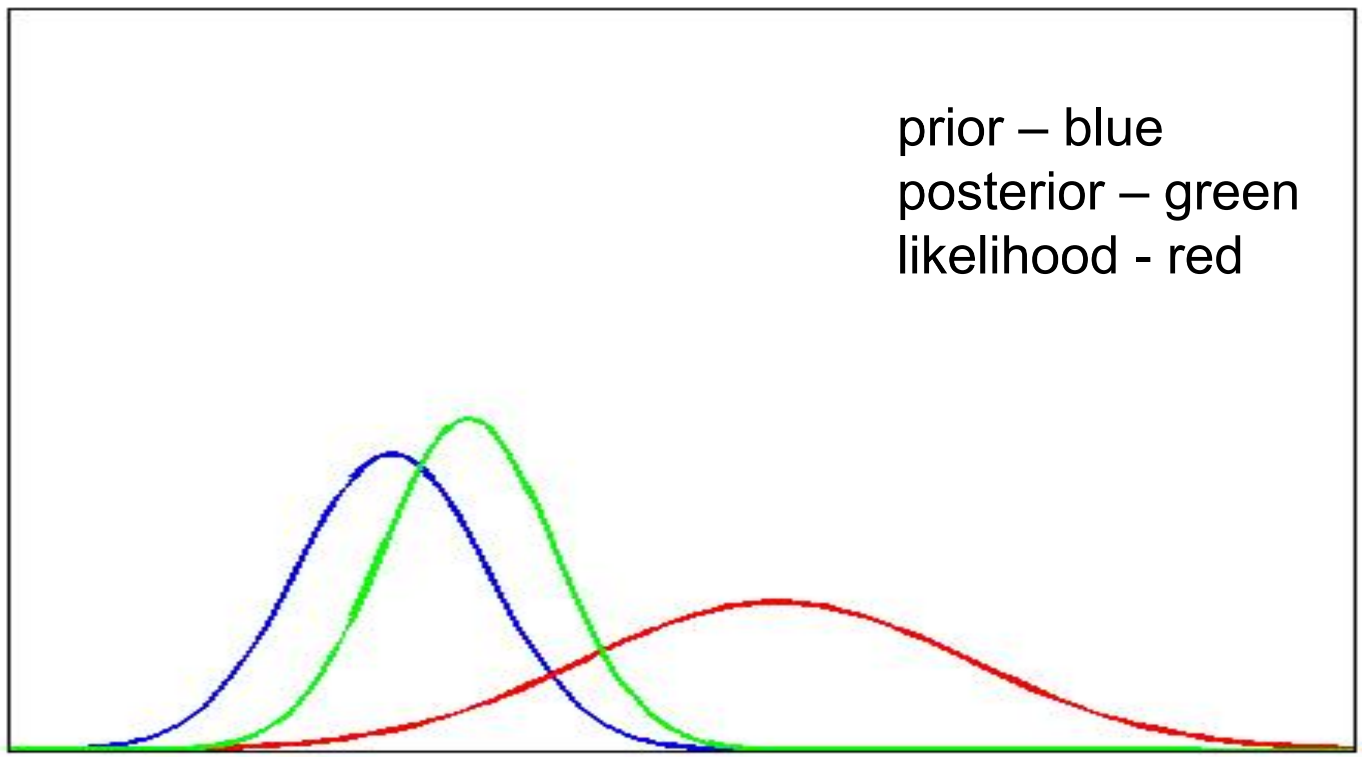
Precision

- The inverse of the variance is called the precision.
- The posterior mean is expressed as a weighted average of the prior mean and the sample mean.
- The weights are proportional to the precisions.
- The posterior mean lies between the prior mean and the sample mean.

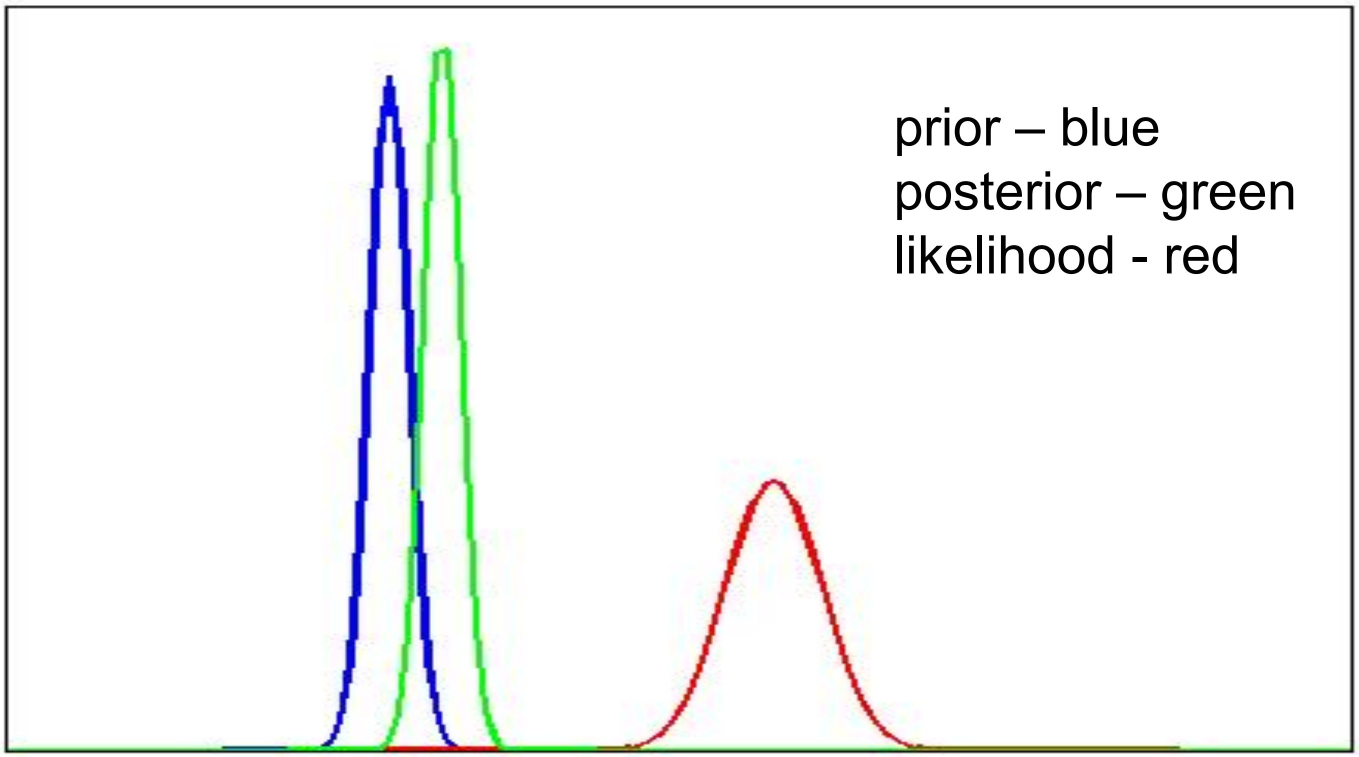
Illustration



Illustration



Illustration



Priors

- In the Bayesian framework the choice of prior is crucial.
- If we have no prior information about the parameters, we may use non-informative priors. These types of priors let the data speak for itself.
- One can also choose the priors in such a way that the posterior lies in the same family of distributions as the prior (conjugate priors).

Posterior Computation

- The posterior distribution is the basis for all Bayesian inference.
- Even if the posterior is known, it can be difficult to obtain exact values of certain posterior quantities (e.g., $E(\theta_1/\theta_2)$).
- By generating random samples from the posterior, all quantities of interest can be approximated using Monte Carlo methods.

Monte Carlo Method

- Let $g(\theta)$ be some function of θ (e.g., $\log(\theta)$).
- Suppose we want to estimate $E(g(\theta)|y)$.
- Generate an i.i.d sequence $\theta_1, \ldots\ldots \theta_N$ from the posterior distribution of θ.
- Estimate $E(g(\theta)|y)$ using $\bar{g} = \frac{1}{N}\sum_{i=1}^{N} g(\theta_i)$

Bayesian Computations

- Sampling from the posterior is effective when it can be implemented.

- However, it is often difficult in practice.

- For most probability distributions there is no simple way to simulate random variables of that particular distribution.

MCMC

- Markov-chain Monte-Carlo (MCMC) is a method for sampling from a posterior distribution.
  - A Markov chain is generated that has the desired distribution as its stationary distribution.
  - The state of the chain after a large number of steps is used as a sample from the desired distribution.
  - Can be extremely computationally expensive.

Variational Bayes

- Variational Bayes (VB) is an approach towards approximating the posterior density which is less computationally intensive than MCMC.
  - Received a lot of attention in fMRI research.
  - It allows one to approximate the posterior density with another density that has a more analytically tractable form.

GLM with Priors

- Consider the standard GLM:

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0,V)$$

- Suppose we place a prior on β, e.g.

$$\beta \sim N(\beta_0, \Sigma_0)$$

- It can be shown that the posterior distribution of β follows a normal distribution. We can use this distribution to perform inference.

- The posterior mean provides a point estimate of β:

$$\hat{\beta} = (X^T V^{-1} X + \Sigma_0^{-1})^{-1} (X^T V^{-1} y + \Sigma_0^{-1} \beta_0)$$

- If Σ_0 large then $\hat{\beta} \approx (X^T V^{-1} X)^{-1} X^T V^{-1} y$

GLS estimate

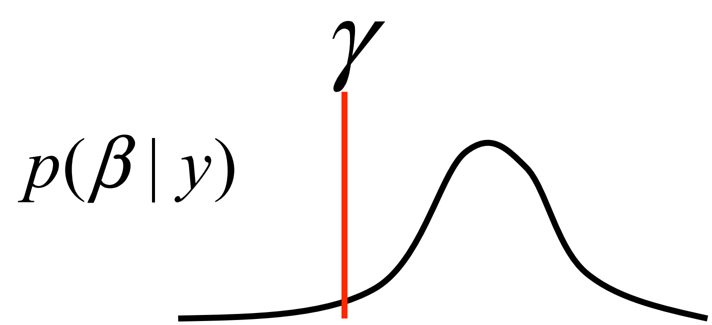
- If β_0=0 then $\hat{\beta} = (X^T V^{-1} X + \Sigma_0^{-1})^{-1} X^T V^{-1} y$

Shrinkage

Posterior Probability Maps

- The Posterior distribution is the probability of getting an effect, given the data p(β|y).

- Posterior probability maps are images of the probability or confidence that an activation exceeds some specified threshold, given the data $p(\beta > \gamma \mid y) > \alpha$



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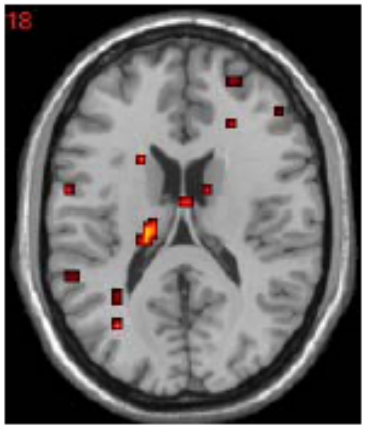


Frequentist

Thresholded t-statistic map (p=0.005, uncorrected)

Bayesian PPM

Voxels with probabilities of task-related increases in activity exceeding α=0.85.

Bayesian Spatial Hierarchical Model [Bowman et al., 2008, *NeuroImage*.]

Comments

- Frequentist and Bayesian methods are answering different questions.
- To combine prior beliefs with data in a principled way use Bayesian inference.
- To construct procedures with guaranteed long run performance use frequentist methods.

Hypothesis Testing

- In classical hypothesis testing we seek to determine whether we can reject a null hypothesis of no effect.
  - The *p*-value is the probability of obtaining a result as or more extreme under the assumption that the null hypothesis is true.
  - We can never accept the null hypothesis.
  - Given enough data every voxel would be significant.

- Bayesian methods allows us to derive probabilities about hypothesis of interest.
  - Not restricted to disproving the null hypothesis.
  - One can argue that it is more interesting to prove a hypothesis, than disprove a hypothesis of no effect.

Multilevel Models

- Data sets where there is a hierarchy of nested populations are often called multilevel.
  - Voxels nested within subjects nested within groups.

- Multilevel models are extensions of regression in which data are structured in groups and coefficients can vary by group.
  - Allows information to be shared across groups.

Multilevel GLM

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, V)$$

p(y|β) = N(X β ,V) represents variability within a subject.

$$\beta = \beta_g + \eta \qquad \eta \sim N(0, \Sigma)$$

p(β|β_g) = N(β_g, Σ) represents variability across subjects

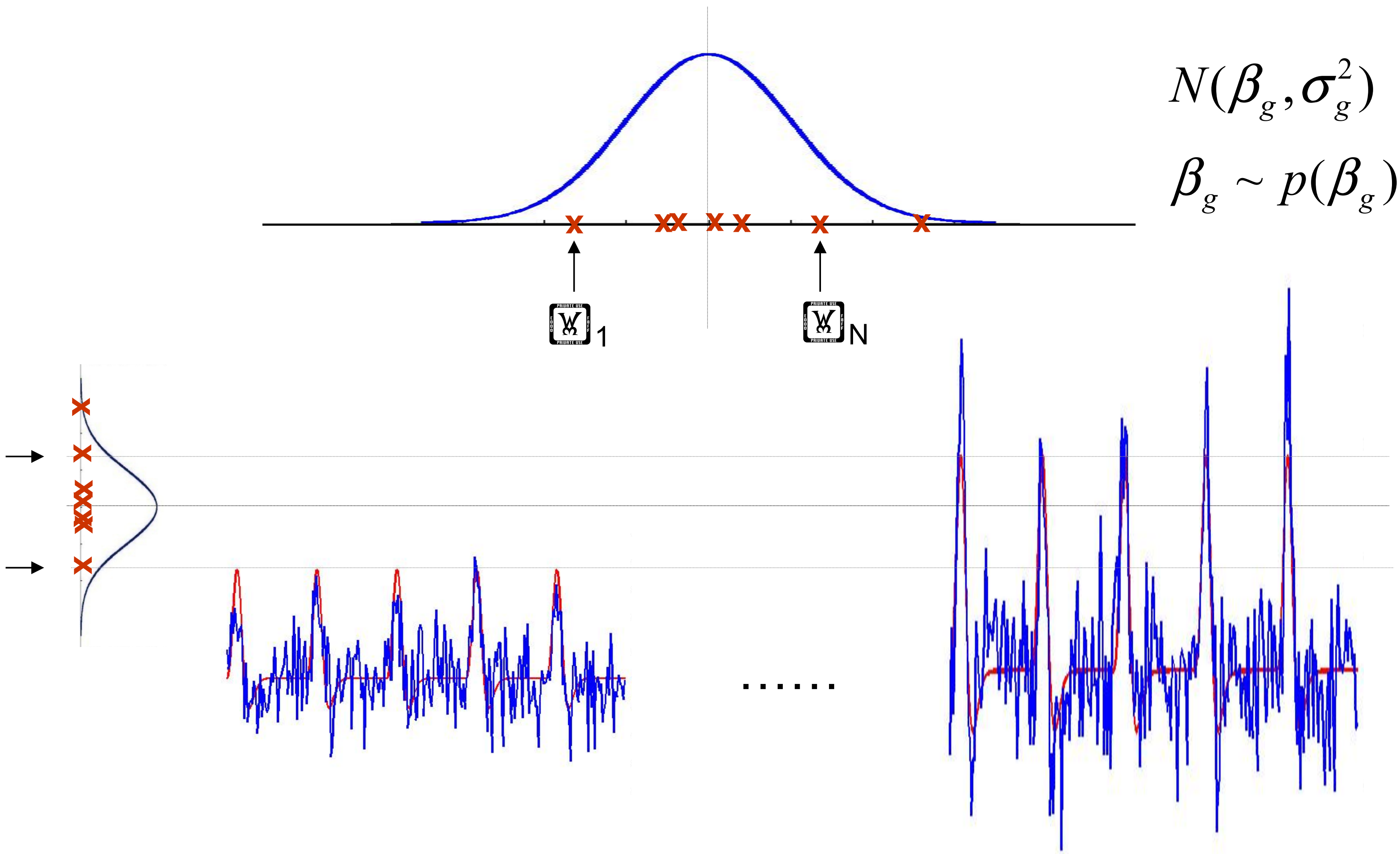
$$\beta_g \sim p(\beta_g)$$

p(β_g) represents information about a fixed but unknown quantity.

Illustration

$$N(\beta_g, \sigma_g^2)$$

$$\beta_g \sim p(\beta_g)$$



Empirical Bayes

- It is common in neuroimaging to use so-called empirical Bayes methods.

- Here the parameters of the prior are estimated directly from the data, rather than being subject to prior specification of their own as is the case in a fully Bayesian model.

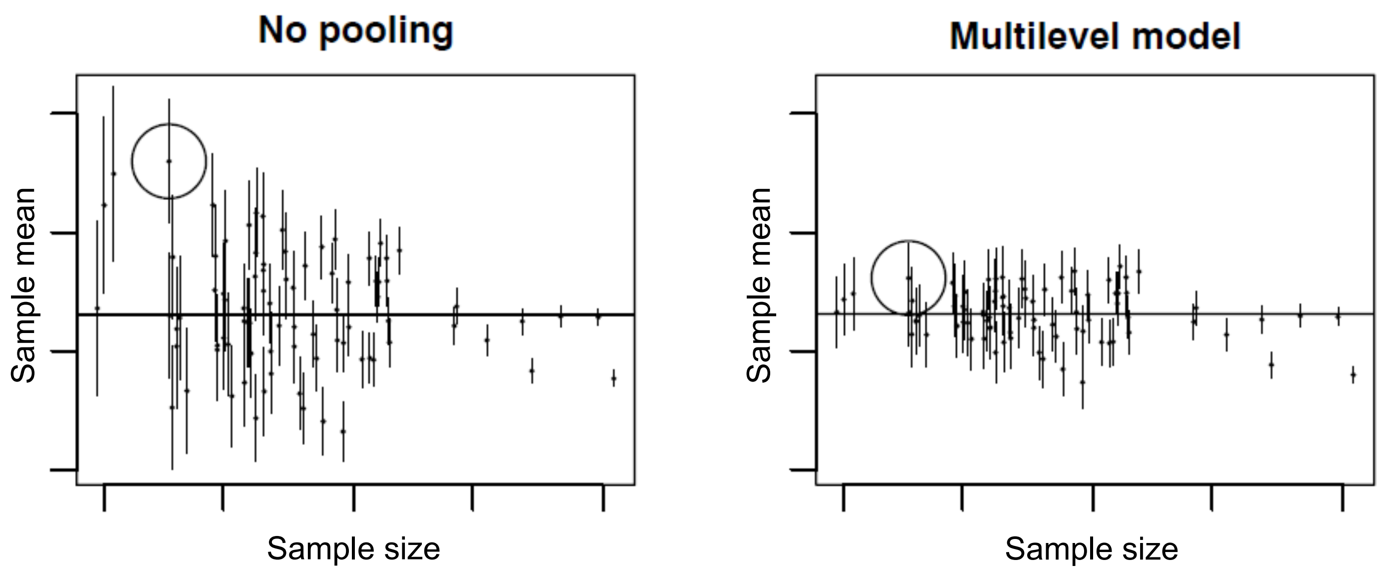
Shrinkage

- Multilevel models allow for heterogeneity across subjects, but still consider values observed in other subjects.

- Each subject-specific estimate gets shrunk towards the overall estimate.
  - The greater the uncertainty, the more shrinkage.
  - The less the uncertainty, the more we trust that individual estimate and the less it gets shrunk.

Example

- Radon levels of houses in 85 counties in Minnesota.



- Most differences no longer statistically significant

Gelman and Hill, 2007

Model Comparison

- Model comparison can be performed to determine whether the data favors one model over another.

- The model evidence is defined as

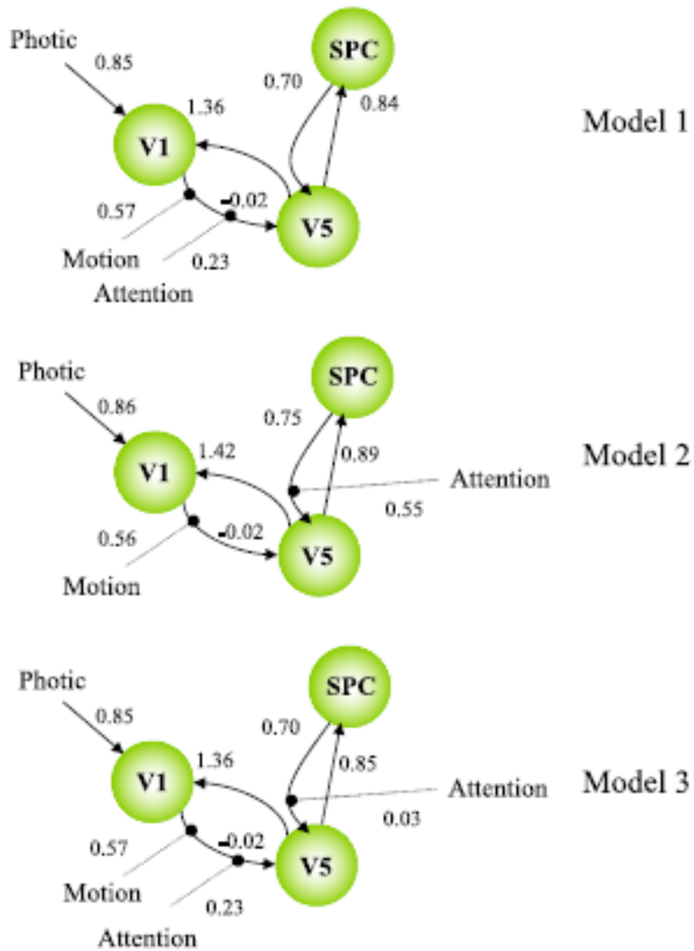
$$p(y \mid m) = \int p(y \mid \theta, m) p(\theta \mid m) d\theta$$

- The Bayes factor for comparing model i to j:

$$B_{ij} = \frac{p(y \mid m = i)}{p(y \mid m = j)}$$

If B_{ij} is large than i more likely than j.

Example



Use Bayes factors to compare three different candidate models.

Table 6
Attention data—comparing modulatory connectivities

	B_{12}	B_{13}	B_{32}
AIC	3.56	2.81	1.27
BIC	3.56	19.62	0.18

Bayes factors provide consistent evidence in favor of the hypothesis embodied in model 1, that attention modulates (solely) the bottom-up connection from V1 to V5. Model 1 is preferred to models 2 and 3.

Penny at al.

Summary

- Bayesian methods are gaining in popularity in fMRI research.
- They allow us to calculate the probability that an activation exceeds some specific threshold, given the data.
  - Not restricted to disproving a null hypothesis.
- Requires specifying prior distributions and can be computationally expensive.